THE SOUND FIELD IN A REVERBERATION ROOM

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May 2011

Note no 31261

CONTENTS

1. INTRODUCTION

Reverberation rooms are important tools in acoustics, used in a variety of standardised measurements, e.g., in measuring the absorption of materials, the sound power of noise sources, and the transmission loss of partitions. They are also used for testing satellite structures at high sound pressure levels.

The study of sound fields in lightly damped enclosed spaces can be approached in, at least, two completely different ways. One can solve the wave equation with the prescribed boundary conditions either analytically or using numerical methods. This approach, which is particularly useful at low frequencies, leads to a description in terms of the modes of the room. Alternatively the problem can be studied using statistical considerations. The second approach, which is particularly appropriate at medium and high frequencies, has the advantage of requiring far less detailed knowledge of the geometry of the room under study, but the resulting model is not very accurate at low frequencies.

The purpose of this note is to give an elementary introduction to these two models. Other important topics in room acoustics such as ray acoustic models, image models, and subjective parameters of importance in design of rooms are *not* dealt with; the reader is referred to the list of books in the bibliography.

2. THE MODAL THEORY OF SOUND IN ENCLOSURES

Sound in lightly damped enclosures will resonate at certain frequencies, the 'natural frequencies'. The sound field in the room at such a resonance frequency is called a 'mode' (or, for reasons that will become apparent, a 'normal mode'), and the spatial distribution of the sound pressure is called the 'mode shape'. If losses at the walls are ignored the Helmholtz equation with the boundary conditions imposed by the rigid walls of the room becomes an eigenvalue problem, and the mathematical solution of the equation leads to eigenfunctions and eigenfrequencies, which are the mathematical terms for the modes and the natural frequencies.

The modes and natural frequencies cannot be determined analytically unless the room is of a simple shape. In rooms of a more complicated geometry the solution must be found using a numerical method, such as for example the finite element method.

2.1 Eigenfrequencies and mode shapes

For simplicity we will concentrate on the particularly simple case of a rectangular room. This means that our task is to find solutions to the Helmholtz equation expressed in the usual Cartesian coordinate system,

$$\frac{\partial^2 \hat{p}}{\partial x^2} + \frac{\partial^2 \hat{p}}{\partial y^2} + \frac{\partial^2 \hat{p}}{\partial z^2} + k^2 \hat{p} = 0 , \tag{2.1}$$

subject to the boundary conditions of zero normal sound pressure gradient at the rigid walls,

$$\frac{\partial \hat{p}}{\partial x} = 0 \text{ at } x = \begin{cases} 0 \\ l_x \end{cases} ; \quad \frac{\partial \hat{p}}{\partial y} = 0 \text{ at } y = \begin{cases} 0 \\ l_y \end{cases} ; \quad \frac{\partial \hat{p}}{\partial z} = 0 \text{ at } z = \begin{cases} 0 \\ l_z \end{cases} . \qquad \text{(2.2a, 2.2b, 2.2c)}$$

It is assumed that the solution to eq. (2.1) can be *factorised*, that is, written as a product of a complex exponential and a function of x, a function of y and a function of z,

$$\hat{p}(x, y, \mathrm{z}, t) = p_x(x) p_y(y) p_z(z) \mathrm{e}^{\mathrm{j}\omega t} . \tag{2.3}$$

Inserting this expression into eq. (2.1) and dividing with $\hat{p}$ yields

$$\frac{1}{p_x(x)} \frac{\mathrm{d}^2 p_x(x)}{\mathrm{d}x^2} + \frac{1}{p_y(y)} \frac{\mathrm{d}^2 p_y(y)}{\mathrm{d}y^2} + \frac{1}{p_z(z)} \frac{\mathrm{d}^2 p_z(z)}{\mathrm{d}z^2} + k^2 = 0. \tag{2.4}$$

A close examination of this equation leads to the conclusion that all terms must be independent of x, y and z. Equating the three first terms of eq. (2.4) with $-k_x^2$, $-k_y^2$ and $-k_z^2$ gives

$$\frac{\mathrm{d}^2 p_x(x)}{\mathrm{d}x^2} + k_x^2 p_x(x) = 0 , \tag{2.5a}$$

$$\frac{\mathrm{d}^2 p_y(y)}{\mathrm{d}y^2} + k_y^2 p_y(y) = 0 , \tag{2.5b}$$

$$\frac{\mathrm{d}^2 p_z(z)}{\mathrm{d}z^2} + k_z^2 p_z(z) = 0 , \tag{2.5c}$$

where the three separation constants are subject to the constraint

$$k_x^2 + k_y^2 + k_z^2 = k^2 . \tag{2.6}$$

Thus eq. (2.1) has been *separated* into three equations, each of which depending on only one coordinate. The three equations have solutions of the form

$$p_x(x) = A\,\mathrm{e}^{-\mathrm{j}k_x x} + B\,\mathrm{e}^{\mathrm{j}k_x x} , \tag{2.7a}$$

$$p_y(y) = C\,\mathrm{e}^{-\mathrm{j}k_y y} + D\,\mathrm{e}^{\mathrm{j}k_y y} , \tag{2.7b}$$

$$p_z(z) = E\,\mathrm{e}^{-\mathrm{j}k_z z} + F\,\mathrm{e}^{\mathrm{j}k_z z} . \tag{2.7c}$$

Consequently, the general solution to eq. (2.1) can be written

$$\hat{p} = \left(A\,\mathrm{e}^{-\mathrm{j}k_x x} + B\,\mathrm{e}^{\mathrm{j}k_x x}\right)\left(C\,\mathrm{e}^{-\mathrm{j}k_y y} + D\,\mathrm{e}^{\mathrm{j}k_y y}\right)\left(E\,\mathrm{e}^{-\mathrm{j}k_z z} + F\,\mathrm{e}^{\mathrm{j}k_z z}\right)\mathrm{e}^{\mathrm{j}\omega t}. \tag{2.8}$$

This expression must satisfy the boundary condition expressed by eq. (2.2). It is easy to see that the boundary conditions at $x = 0$, $y = 0$ and $z = 0$ imply that $B = A$, $D = C$ and $F = E$, from which it follows that

$$p_x\left(x\right) = 2A\cos(k_x x), \quad p_y\left(y\right) = 2C\cos(k_y y), \quad p_z\left(z\right) = 2E\cos(k_z z). \qquad \text{(2.9a, 2.9b, 2.9c)}$$

The boundary conditions at $x = l_x$, $y = l_y$ and $z = l_z$ are satisfied only for the following discrete values of k_x, k_y and k_z,

$$k_x l_x = n_x \pi, \qquad k_y l_y = n_y \pi, \qquad k_z l_z = n_z \pi, \qquad \text{(2.10a, 2.10b, 2.10c)}$$

where n_x, n_y and n_z are integers. All in all we can write the sound pressure as a sum of terms of the form

$$\psi_N(x,y,z) = \Lambda_N \cos\left(\frac{n_x \pi x}{l_x}\right)\cos\left(\frac{n_y \pi y}{l_y}\right)\cos\left(\frac{n_z \pi z}{l_z}\right), \qquad (2.11)$$

that is,

$$\hat{p}(x,y,z,t) = \sum_N A_N \psi_N(x,y,z) e^{j\omega t}, \qquad (2.12)$$

where N represents the three integers n_x, n_y and n_z,

$$\sum_N = \sum_{n_x=0}^{\infty} \sum_{n_y=0}^{\infty} \sum_{n_z=0}^{\infty}. \qquad (2.13)$$

The modal amplitudes, A_N, depend on the position and strength of the source; see section 2.3. The factor

$$\Lambda_N = \sqrt{\varepsilon_{n_x} \varepsilon_{n_y} \varepsilon_{n_z}}, \qquad (2.14)$$

in which $\varepsilon_0 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 \ldots = 2$, is a normalisation constant. Combining eqs. (2.6) and (2.10) gives us an expression for the natural frequencies of the room,[1]

$$\omega_N = k_N c = c\left(\left(\frac{n_x \pi}{l_x}\right)^2 + \left(\frac{n_y \pi}{l_y}\right)^2 + \left(\frac{n_z \pi}{l_z}\right)^2\right)^{1/2}. \qquad (2.15)$$

Each term in the sum given by eq. (2.12) represents a mode. In the special case where all three indices are zero we have the fundamental cavity mode in which the sound pressure is independent of the position in the room and the air in the room acts like a spring (see example 2.1). If two out of the three indices are zero, we have an *axial mode* with wave motion in just one direction; two-dimensional modes, for which one index equals zero, are known as *tangential modes*; and three-dimensional modes are also called *oblique modes*. In figure 2.1 are shown contours of equal sound pressure in two different tangential modes. Note the nodal planes in which the sound pressure is zero.

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[1] If the dimensions of the room are commensurable (say, $l_x = 2l_y = 3l_z$) some of the natural frequencies will coincide. As a result the frequency response of the room will be more irregular. This phenomenon, which is called modal degeneracy, should be avoided. The dimensions of well-designed rectangular reverberation rooms (or loudspeaker enclosures) are not related by whole numbers.

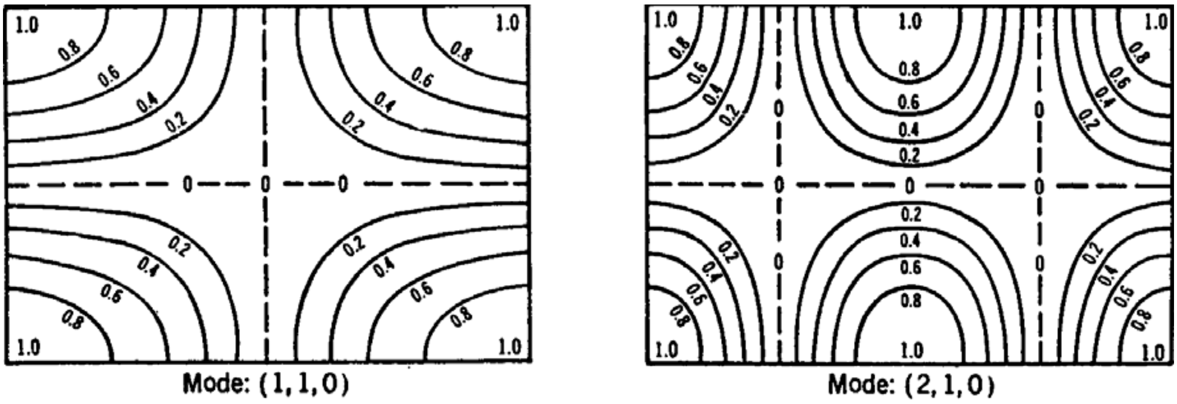

Figure 2.1 Equal pressure contours of the (1, 1, 0) mode and the (2, 1, 0) mode in a rectangular room. (From ref. [1]).

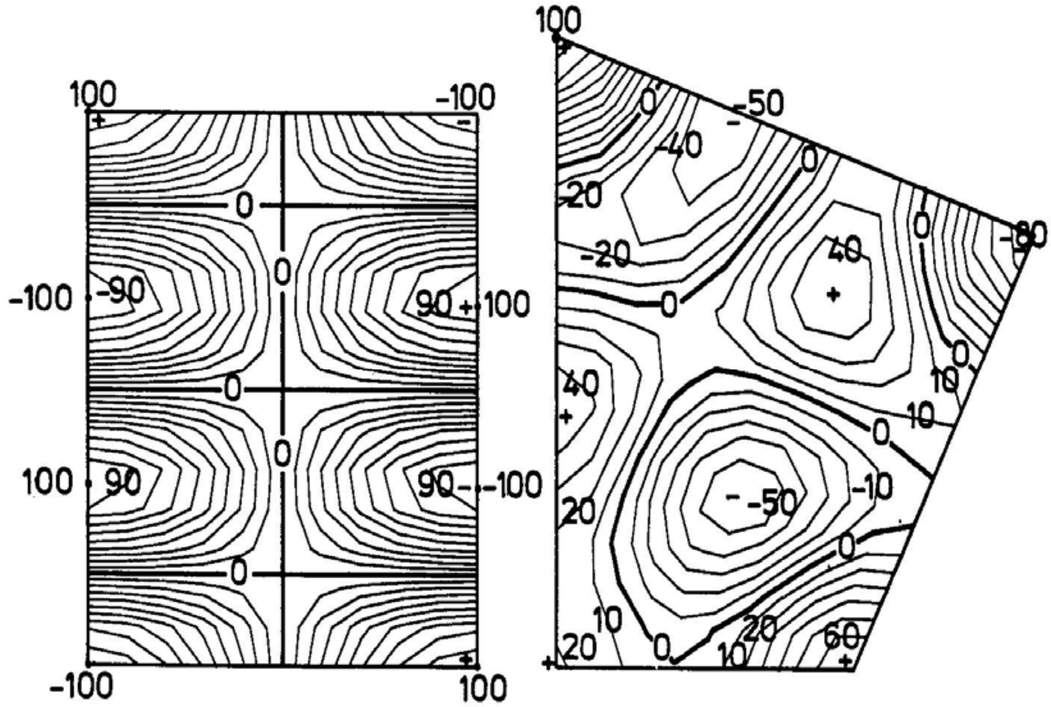

Figure 2.2 Modal patterns in a rectangular and in a nonrectangular room. (From ref. [2].)

Modes in a room may be interpreted as sums of interfering waves travelling in various directions. For example, if we rewrite the expression for an axial mode in the x-direction as a sum of two complex exponentials,

$$\psi_N(x, y, z) = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{j}k_x x} + \mathrm{e}^{\mathrm{j}k_x x}\right), \tag{2.16}$$

it becomes apparent that it may be regarded as a sum of two plane waves with the same amplitude, one travelling in the positive x-direction and one travelling in the opposite direction – in other words a standing wave. The two waves interfere in such a manner that the boundary conditions at $x = 0$ and $x = l_x$ are satisfied; this is the case only for the discrete values of k_x

given by eq. (2.10a). In the same way we can write the general expression for a three-dimensional mode in the form

$$
\begin{aligned}
\Psi_N(x,y,z) &= \frac{1}{\sqrt{8}}\left(\mathrm{e}^{-\mathrm{j}k_x x}+\mathrm{e}^{\mathrm{j}k_x x}\right)\left(\mathrm{e}^{-\mathrm{j}k_y y}+\mathrm{e}^{\mathrm{j}k_y y}\right)\left(\mathrm{e}^{-\mathrm{j}k_z z}+\mathrm{e}^{\mathrm{j}k_z z}\right) \\
&= \frac{1}{\sqrt{8}}\left(\mathrm{e}^{-\mathrm{j}(k_x x+k_y y+k_z z)}+\mathrm{e}^{-\mathrm{j}(k_x x+k_y y-k_z z)}+\mathrm{e}^{-\mathrm{j}(k_x x-k_y y+k_z z)}+\mathrm{e}^{-\mathrm{j}(k_x x-k_y y-k_z z)}\right. \\
&\quad \left.+\mathrm{e}^{\mathrm{j}(k_x x+k_y y+k_z z)}+\mathrm{e}^{\mathrm{j}(k_x x+k_y y-k_z z)}+\mathrm{e}^{\mathrm{j}(k_x x-k_y y+k_z z)}+\mathrm{e}^{\mathrm{j}(k_x x-k_y y-k_z z)}\right),
\end{aligned} \tag{2.17}
$$

which shows that an oblique mode can be decomposed into eight interfering plane waves that propagate in the directions (k_x, k_y, k_z), $(k_x, k_y, \text{-}k_z)$, $(k_x, \text{-}k_y, k_z)$, $(k_x, \text{-}k_y, \text{-}k_z)$, $(\text{-}k_x, \text{-}k_y, \text{-}k_z)$, $(\text{-}k_x, \text{-}k_y, k_z)$, $(\text{-}k_x, k_y, \text{-}k_z)$ and $(\text{-}k_x, k_y, k_z)$. These plane waves may be interpreted as one obliquely incident plane wave and its reflections from the walls of the room. The eight waves interfere in such a manner that the boundary conditions (eqs. (2.2)) are satisfied.

It should finally be emphasised that the sound field in an enclosure of *any* shape can be decomposed into modes, although simple analytical solutions are available only for rectangular, cylindrical and spherical enclosures. An example of a mode in an irregular room is given in figure 2.2. In such a room there are no nodal planes but curved nodal surfaces.

2.2 The modal density

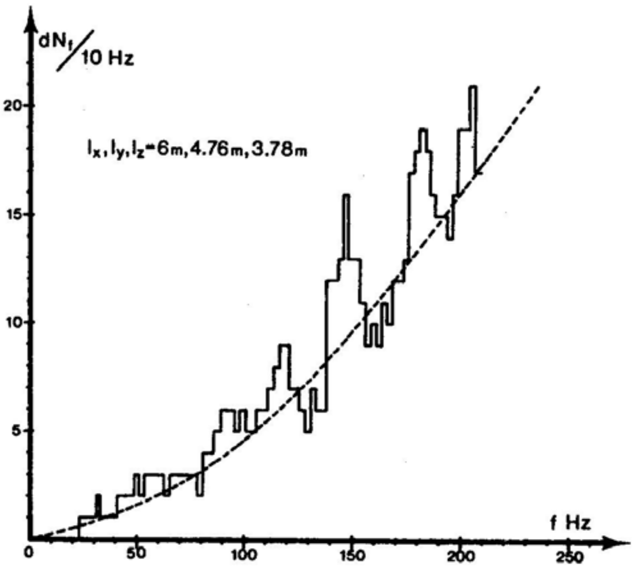

Figure 2.3 Number of natural frequencies in a 10-Hz band compared with the smoothed modal density. (From ref. [3]).

The exact distribution of the eigenfrequencies of a rectangular room depends on the dimensions as indicated by eq. (2.15). The modal density, that is, the average number of modes per unit bandwidth, is in general a somewhat irregular function of the frequency; see figure 2.3. However, it is possible to derive a smoothed expression for the modal density, as follows. From the observation that the natural frequencies of the axial modes in the x-direction are equidistantly distributed on the frequency axis,

$$\frac{c}{2l_x}, \frac{c}{l_x}, \frac{3c}{2l_x}, \frac{2c}{l_x}, \ldots \tag{2.18}$$

we conclude that there are about $f/(c/2l_x)$ such modes below the frequency f. In a similar manner we can derive an approximate expression for the number of tangential *x-y* modes below the frequency f by counting the number of rectangles with dimensions ($c/2l_x$, $c/2l_y$) in a quarter of a circle with radius f (see figure 2.4). The result is

$$N_{xy}(f) \simeq \frac{\frac{1}{4}\pi f^2}{\frac{c}{2l_x}\frac{c}{2l_y}} = \frac{\pi l_x l_y}{c^2} f^2 . \tag{2.19}$$

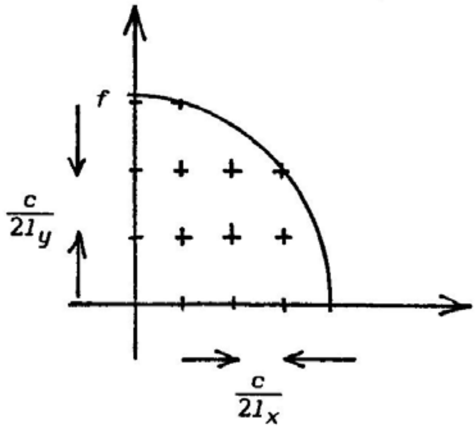

Figure 2.4 Counting the number of x-y modes below the frequency f.

Finally we can derive an approximate expression for the number of three-dimensional modes below f by counting the number of boxes with dimensions ($c/2lx$, $c/2ly$, $c/2lz$) in an eighth of a sphere with the radius f,

$$N(f) \simeq \frac{\frac{1}{8}\frac{4}{3}\pi f^3}{\frac{c}{2l_x}\frac{c}{2l_y}\frac{c}{2l_z}} = \frac{4\pi V}{3c^3} f^3 . \tag{2.20}$$

The observation that the number of one-, two-, and three-dimensional modes below f is proportional to f, f^2 and f^3 leads to the conclusion that three-dimensional modes will dominate except at low frequencies; therefore we will ignore axial and tangential modes. Differentiating eq. (2.20) with respect to the frequency gives the modal density,

$$n(f) = \frac{\mathrm{d}N(f)}{\mathrm{d}f} \simeq \frac{4\pi V}{c^3} f^2 . \tag{2.21}$$

It can be shown that this expression is asymptotically valid in any room, irrespective of its shape [4].

2.2 The Green's function

An important property of the modes in a room can be deduced as follows. Since every mode satisfies the Helmholtz equation, we can write

$$\nabla^2\psi_m + k_m^2\psi_m = 0, \qquad \nabla^2\psi_n + k_n^2\psi_n = 0, \tag{2.22a, 2.22b}$$

and therefore,

$$\psi_n(\nabla^2 + k_m^2)\psi_m - \psi_m(\nabla^2 + k_n^2)\psi_n = 0. \tag{2.23}$$

However, this equation can be rewritten in the form

$$\nabla \cdot (\psi_n \nabla \psi_m - \psi_m \nabla \psi_n) + (k_m^2 - k_n^2)\psi_m\psi_n = 0. \tag{2.24}$$

Integrating over the volume of the room and applying Gauss's theorem[2] on the first term gives

$$\int_S (\psi_n \nabla \psi_m - \psi_m \nabla \psi_n) \cdot \mathrm{d}S + (k_m^2 - k_n^2)\int_V \psi_m\psi_n \mathrm{d}V = 0, \tag{2.25}$$

where S is the surface of the room. At the rigid walls the normal component of the gradient of any eigenfunction is zero, and therefore the left-hand term is zero. It follows that the eigen-functions are *orthogonal*, that is, that

$$\int_V \psi_m\psi_n \mathrm{d}V = 0, \tag{2.26}$$

unless $m = n$; hence the term 'normal mode'. It is customary to normalise the eigenfunctions so that

$$\frac{1}{V}\int_V \psi_m\psi_n \mathrm{d}V = \delta_{mn}, \tag{2.27}$$

which implies that

$$\int_V \psi_m^2 \mathrm{d}V = V. \tag{2.28}$$

Note that these considerations have *not* been limited to the special case of a rectangular room.

It is interesting to study how a source placed at a certain position in the room will excite the various modes. The Green's function for the sound field in a room with rigid walls[3]

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[2] According to Gauss's theorem the volume integral of the divergence of a vector $\boldsymbol{\Xi}$ is identical with the corresponding surface integral of the normal component of the vector, that is,

$$\int_V \nabla \cdot \boldsymbol{\Xi} \mathrm{d}V = \int_S \boldsymbol{\Xi} \cdot \mathrm{d}\mathbf{S}.$$

[3] The Green's function is the sound pressure at one point, $\mathbf{r}$, generated by a (normalised) point source with frequency independent volume acceleration at another point, $\mathbf{r}_0$. See e.g. ref. [5].

can be derived as follows. We are looking for solutions to the inhomogeneous Helmholtz equation

$$\nabla^2 G(\mathbf{r},\mathbf{r_0}) + k^2 G(\mathbf{r},\mathbf{r_0}) = -\delta(\mathbf{r}-\mathbf{r_0}) \tag{2.29}$$

with the boundary condition

$$\frac{\partial G(\mathbf{r},\mathbf{r_0})}{\partial n} = 0 \tag{2.30}$$

on the walls. Any sound field in the room can be expressed in terms of the modes of the room, that is, functions that satisfy the equation

$$\nabla^2 \psi_m(\mathbf{r}) + k_m^2 \psi_m(\mathbf{r}) = 0 \tag{2.31}$$

and the boundary condition mentioned above. Therefore,

$$G(\mathbf{r},\mathbf{r_0}) = \sum_m A_m \psi_m(\mathbf{r}). \tag{2.32}$$

The source term (the right-hand side of eq. (2.29)) can also be expanded into a sum of modes,

$$-\delta(\mathbf{r}-\mathbf{r_0}) = \sum_m B_m \psi_m(\mathbf{r}). \tag{2.33}$$

Multiplying with ψ_n and integrating over the volume of the room gives, if we make use of the fact that the modes are orthogonal (cf. eq. (2.27)),

$$-\int_V \delta(\mathbf{r}-\mathbf{r_0})\psi_n(\mathbf{r})\mathrm{d}V = -\psi_n(\mathbf{r_0}) = \int_V \sum_m B_m \psi_m(\mathbf{r})\psi_n(\mathbf{r})\mathrm{d}V = B_n V, \tag{2.34}$$

which shows that

$$B_m = -\frac{\psi_m(\mathbf{r_0})}{V}, \tag{2.35}$$

and thus

$$-\delta(\mathbf{r}-\mathbf{r_0}) = -\frac{1}{V}\sum_m \psi_m(\mathbf{r})\psi_m(\mathbf{r_0}). \tag{2.36}$$

It now follows that

$$\begin{aligned}(\nabla^2 + k^2)G(\mathbf{r},\mathbf{r_0}) &= (\nabla^2 + k^2)\sum_m A_m \psi_m(\mathbf{r}) = \sum_m A_m \left(\nabla^2 + k_m^2 - k_m^2 + k^2\right)\psi_m(\mathbf{r}) \\ &= \sum_m A_m \left(k^2 - k_m^2\right)\psi_m(\mathbf{r}) = -\frac{1}{V}\sum_m \psi_m(\mathbf{r})\psi_m(\mathbf{r_0}),\end{aligned} \tag{2.37}$$

from which we deduce that

$$A_m = -\frac{1}{V}\frac{\psi_m(\mathbf{r_0})}{k^2 - k_m^2}. \tag{2.38}$$

Finally we can write the Green's function as

$$G(\mathbf{r},\mathbf{r_0}) = -\frac{1}{V}\sum_m \frac{\psi_m(\mathbf{r})\psi_m(\mathbf{r_0})}{k^2 - k_m^2}. \tag{2.39}$$

Note the symmetry with respect to source and receiver position, in agreement with the reciprocity principle. A point source placed on a nodal surface of a given mode does not excite the mode. Note also that each mode, not surprisingly, contributes most to the sound field when driven near its natural frequency ($k \simeq k_m$). It can be seen that the response is unlimited if the frequency of the excitation coincides with one of the natural frequencies.

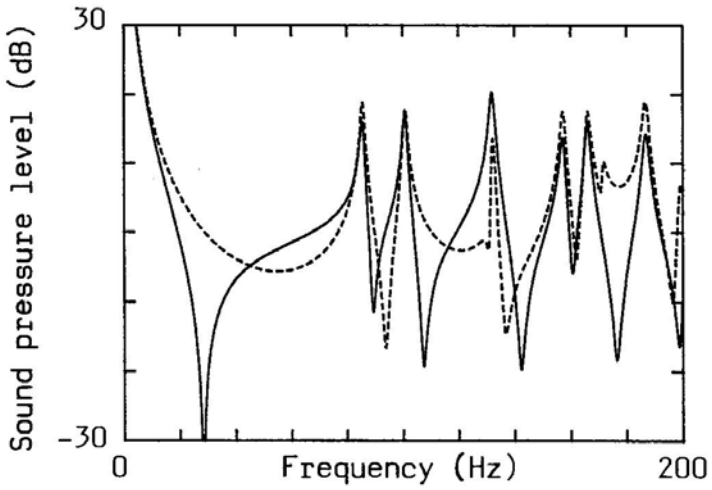

Figure 2.5 Magnitude of the Green's function in a room as a function of the frequency. Solid line: source and receiver points close to each other; dashed line: source and receiver points far from each other.

In practice there are, of course, losses in any room, even in a room with walls of solid concrete.[4] As a result the eigenvalues of the problem become complex, with small imaginary parts equal to $1/(2\tau_m c)$, where τ_m is the time constant of the m'th mode (see section 4.1). This leads to the expression

$$G(\mathbf{r},\mathbf{r_0}) \simeq -\frac{1}{V}\sum_m \frac{\psi_m(\mathbf{r})\psi_m(\mathbf{r_0})}{k^2 - k_m^2 - \mathrm{j}k_m/(\tau_m c)} \simeq -\frac{1}{V}\sum_m \frac{\psi_m(\mathbf{r})\psi_m(\mathbf{r_0})}{k^2 - k_m^2 - \mathrm{j}k/(\tau_m c)}\ , \tag{2.40}$$

[4] Even in a room with perfectly rigid walls there are some losses because of the viscosity of air and because the process of sound tends to be isothermal rather than adiabatic in a thin boundary layer near every wall. To a rough approximation the effect of the thermal and viscous losses at the walls in a reverberation room can be described in terms of a finite absorption coefficient,

$$\alpha \simeq 1.8 \cdot 10^{-4}\sqrt{f},$$

where f is the frequency in hertz (Cremer and Müller, 1982). See chapter 4 for relations between the absorption coefficient, the wall admittance, the time constant, the reverberation time, etc.

where the second approximation has the advantage over the first one that it corresponds to a real-valued, causal time function.[5] The left-hand version corresponds to complex eigenvalues caused by losses at the boundaries; the right-hand version corresponds to a *medium* with losses.

It can be seen from eq. (2.40) that the response of any mode is limited also when it is driven at its natural frequency. It is also easy to show that the 3-dB bandwidth of the m'th mode (in hertz) is $1/(2\pi\tau_m)$. We will study the effect of losses in chapter 4.

Figure 2.5 shows two examples of the eq. (2.40) plotted as a function of the frequency, and figure 2.6 shows two examples of eq. (2.40) plotted as a function of the distance between the source and the receiver position.

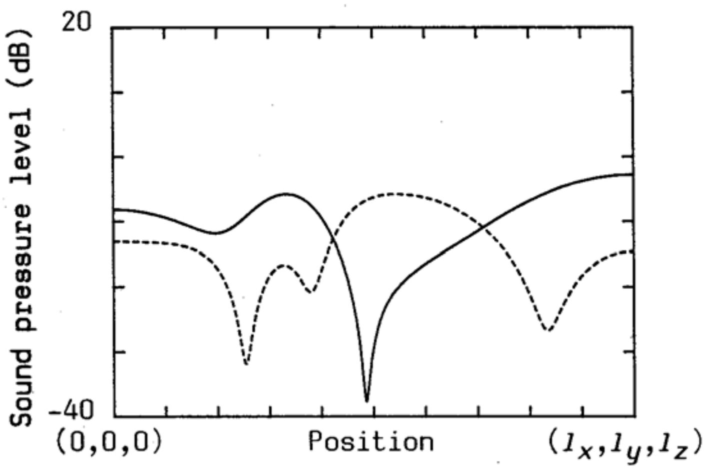

Figure 2.6 Magnitude of the Green's function in a room as a function of the receiver position. The room is driven close to a natural frequency (solid line) and midway between two natural frequencies (dashed line).

Example 2.1

If the room is driven at a very low frequency the response will be dominated by the (0, 0, 0) mode. Thus eq. (2.40) becomes,

$$G(\mathbf{r},\mathbf{r_0}) \simeq -\frac{1}{Vk^2} = -\frac{c^2}{V\omega^2},$$

which is independent of $\mathbf{r}$ and $\mathbf{r}_0$. The sound pressure generated by a monopole with the volume velocity $Q\mathrm{e}^{\mathrm{j}\omega t}$ is obtained by multiplying with $\mathrm{j}\omega\rho Q$ [5],

$$\hat{p}(\mathbf{r}) = \mathrm{j}\omega\rho Q G(\mathbf{r},\mathbf{r_0})\mathrm{e}^{\mathrm{j}\omega t} \simeq \frac{\rho c^2}{\mathrm{j}\omega V} Q\mathrm{e}^{\mathrm{j}\omega t},$$

and this is seen to agree with the fact that the acoustic impedance of a cavity with dimensions much shorter than the wavelength is

$$Z_a = \frac{\rho c^2}{\mathrm{j}\omega V} = \frac{\gamma p_0}{\mathrm{j}\omega V},$$

where γ is the ratio of specific heats and p_0 is the static pressure.

[5] A realistic frequency response should correspond to a real-valued, causal impulse response. This implies that the real part of the frequency response must be an even function of the frequency, the imaginary part should be an odd function of the frequency, and that the real and imaginary part are related by the Hilbert transform [6, 7].

The (0, 0, 0) ‘cavity mode’ is responsible for the low frequency boost that can be heard in small spaces, e.g., inside cars. The acoustic impedance of the cavity is much larger than the radiation impedance in the open, and therefore even a small loudspeaker can generate a surprisingly high sound pressure level above its resonance frequency and up to, say, 80 Hz in a car.

Example 2.2

It is interesting to compare eq. (2.39) with the free space Green’s function,

$$G(\mathbf{r},\mathbf{r_0}) = \frac{\mathrm{e}^{-\mathrm{j}kR}}{4\pi R},$$

where $R = |\mathbf{r} - \mathbf{r}_0|$, and with the Green’s function for a semi-infinite lossless duct [5],

$$G(\mathbf{r},\mathbf{r_0}) = -\frac{\mathrm{j}}{S}\sum_m \frac{\psi_m(\mathbf{r})\psi_m(\mathbf{r_0})}{\sqrt{k^2 - k_m^2}}\mathrm{e}^{-\mathrm{j}k_{zm}z}.$$

There are evidently strong similarities between modes in ducts and rooms, but the cut-off phenomenon that occurs when a duct mode is driven below a certain frequency is peculiar to ducts.

At very low frequencies a monopole with the volume velocity $Q\mathrm{e}^{\mathrm{j}\omega t}$ will generate the sound pressure

$$\hat{p}(\mathbf{r}) = \begin{cases} \dfrac{\mathrm{j}\omega\rho Q}{4\pi R}\mathrm{e}^{\mathrm{j}(\omega t - kR)} & \text{in free space,} \\ \dfrac{\rho c Q}{S}\mathrm{e}^{\mathrm{j}(\omega t - kz)} & \text{in a semi-infinite duct,} \\ \dfrac{\rho c^2 Q}{\mathrm{j}\omega V}\mathrm{e}^{\mathrm{j}\omega t} & \text{in a room.} \end{cases}$$

Note that the radiation impedance in free space is mass-like, that the radiation impedance in the duct is real-valued at low frequencies where only plane waves can propagate, and that the air in a room at low frequencies behaves like an elastic spring.

3. AN INTRODUCTION TO STATISTICAL ROOM ACOUSTICS

In theory the validity of the modal expressions derived in chapter 2 is not restricted to low frequencies. It may seem surprising, therefore, that a completely different approach based on statistical considerations is actually more useful at medium and high frequencies than the deterministic approach described in the foregoing. Evidently, a model based on statistical considerations can only give statistical answers; we may, for example, be able to predict with a certain level of confidence that the sound pressure level is within a certain range. However, we can never make deterministic predictions with a probabilistic model. By contrast, eq. (2.40) appears to be exact. Why the concern with statistical models, then?

There are two reasons. One reason is that expressions based on sums of modes are in practice less useful at high frequencies than they would seem to be. The problem is that when hundreds of complex terms are summed the result becomes very sensitive to small errors in each term, and such errors are likely to occur. For example, the dimensions of the room might be slightly different from the dimensions used in the model, and the temperature and thus the speed of sound may differ a bit from the value used in modelling the room. Even very small modelling errors will shift the natural frequencies of the modes, and the amplitude and phase of each of the terms that correspond to modes driven near their natural frequency may change somewhat. As a result the sum can be completely wrong at any given frequency.

The other reason is that statistical models, as we shall see, can be surprisingly powerful in the sense that they make it possible to predict a number of characteristics of, e.g., the sound field in a reverberation room on the basis of very little information. Indeed, the statistical properties of the sound field in a reverberation room driven with a pure tone can be predicted in considerable detail without *any* knowledge of the room.

Only some of the most fundamental results of the statistical model will be presented here. The reader is referred to Pierce (1989) and refs. [8, 9] for further details.

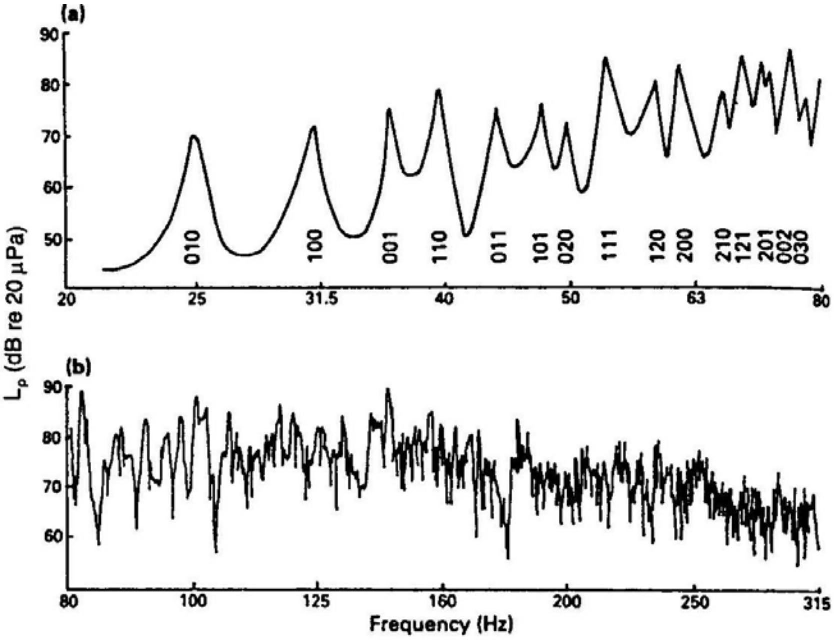

Figure 3.1 A typical frequency response in a room. (From ref. [10].)

The background for the statistical theory is the fact that the modal density in a room to a good approximation increases with the square of the frequency, irrespective of the shape of the room, as we have seen in chapter 2,

$$n(f) \simeq \frac{4\pi V}{c^3} f^2 \tag{3.1}$$

(see figure 3.1). Combined with the finite 3-dB bandwidth of the modes,

$$\Delta f = \frac{1}{2\pi\tau} \simeq \frac{cA}{8\pi V} \simeq \frac{3\ln 10}{\pi T_{60}} \simeq \frac{2.2}{T_{60}}, \tag{3.2}$$

where T_{60} is the reverberation time and A is the total absorption area of the room (see sections 4.2 and 5.1), this leads to the following expression for the modal overlap,

$$M = n(f)\Delta f \simeq \frac{A}{2c^2} f^2 \simeq \frac{12\ln 10 V}{T_{60}c^3} f^2 \simeq \frac{27.6V}{T_{60}c^3} f^2. \tag{3.3}$$

The modal overlap is the average number of modes excited by a pure tone under the simplifying assumption that each mode is excited only if the frequency of the pure tone is within a band of the width $2.2/T_{60}$ centred at the natural frequency of the mode.[6] The frequency

$$f_s = 2000\sqrt{\frac{T_{60}}{V}} \tag{3.4}$$

(with T_{60} in seconds and V in cubic metres) is known as the Schroeder frequency[7] or 'Schroeder's large room frequency'. The Schroeder frequency is the frequency above which there is sufficient modal overlap to justify a statistical approach (via the central-limit theorem) even for pure-tone excitation. Above this frequency the modal overlap exceeds (approximately) three, and experience shows that this is sufficient to justify the statistical approach [12].[8]

3.1 The perfectly diffuse sound field

A particularly simple statistical model of the sound field in a reverberation room assumes that the sound field is 'perfectly diffuse'. The perfectly diffuse sound field is composed of sound waves coming from all directions. This leads to the concept of a sound field in an unbounded medium generated by distant, uncorrelated sources of random noise evenly distributed over all directions. Since the sources are uncorrelated there would not be interference phenomena in such a sound field, and the field would therefore be completely homogeneous and isotropic. For example, the sound pressure level would be the same at all positions, and temporal correlation functions between linear quantities measured at two points would depend only on the distance between the two points. The time-averaged sound intensity would be zero at all positions. An approximation to this perfectly diffuse sound field might be generated by a number of loudspeakers driven with uncorrelated noise in a large anechoic room. The sound field in a reverberation room driven by one single source is quite different, of course. Nevertheless, combined with simple energy balance considerations the assumption that the sound field in a reverberation room is perfectly diffuse leads to an extremely useful relation between the sound power emitted by a source of noise in the room, the total absorption of the room, and the sound pressure generated by the source.

The derivation is fairly simple. It is assumed that the sound field in a reverberation room is composed of incoherent plane waves of the form

$$\hat{p}(\theta,\varphi) = \frac{C}{\sqrt{4\pi}} \mathrm{e}^{\mathrm{j}(\omega t - k_x x - k_y y - k_z z)}. \tag{3.5}$$

All waves have an amplitude of $C/\sqrt{4\pi}$, and they arrive from all directions, given by the three components of the wavenumber vector,

$$k_x = k\sin\theta\cos\varphi, \quad k_y = k\sin\theta\sin\varphi, \quad k_z = k\cos\theta. \qquad \text{(3.6a, 3.6b, 3.6c)}$$

This model is based on the assumption that interference between the various waves can be ignored, which leads to a uniform mean-square pressure of

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[6] Note that the losses of a reverberation room and thus T_{60} depend on the frequency. However, the reverberation time is usually a slowly varying function of the frequency. It is often measured in one-third octave bands.

[7] This concept is named after M.R. Schroeder, who derived fundamental parts of the statistical theory in a paper published in 1954 [11].

[8] With a modal overlap of three, the sound field in a rectangular room is essentially composed of 24 plane waves (cf. eq. (2.17)).

$$p_{\text{rms}}^2 = \frac{|C|^2}{2\cdot 4\pi}\int_{-\pi}^{\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta = \frac{|C|^2}{2}, \tag{3.7}$$

except, as we shall see later, near the walls. The z-component of the particle velocity in a single wave is

$$\hat{u}_z(\theta,\varphi) = \frac{C}{\rho c\sqrt{4\pi}}\frac{k_z}{k}\mathrm{e}^{\mathrm{j}(\omega t - k_x x - k_y y - k_z z)} = \frac{C\cos\theta}{\rho c\sqrt{4\pi}}\mathrm{e}^{\mathrm{j}(\omega t - k_x x - k_y y - k_z z)}, \tag{3.8}$$

and the corresponding sound intensity is

$$I_z(\theta,\varphi) = \tfrac{1}{2}\,\mathrm{Re}\left\{\hat{p}(\theta,\varphi)\hat{u}_z^*(\theta,\varphi)\right\} = \frac{|C|^2}{8\pi\rho c}\cos\theta. \tag{3.9}$$

If this is integrated over the full solid angle of 4π the result is zero; there is no net intensity in any direction in the perfectly diffuse sound field. However, the boundaries of the room are exposed to a finite *incident* sound intensity. For example, the sound power incident per unit area on a wall at $z = l_z$ is found by restricting the sound incidence to a hemisphere,

$$I_{\text{inc},z} = \frac{|C|^2}{8\pi\rho c}\int_{-\pi}^{\pi}\mathrm{d}\varphi\int_0^{\pi/2}\cos\theta\sin\theta\,\mathrm{d}\theta = \frac{|C|^2}{8\rho c} = \frac{p_{\text{rms}}^2}{4\rho c}, \tag{3.10}$$

Note that this is four times less than in a plane wave of normal incidence.

The losses of the walls are described in terms of the absorption coefficient (or absorption factor) of the walls, α. This is the absorbed fraction of the incident sound power.[9] Obviously, the absorption coefficient of any material must take values between naught and unity ($0 \le \alpha \le 1$). The total absorption of the room, A, is calculated by multiplying each area with its absorption coefficient,

$$A = \sum_i S_i\alpha_i. \tag{3.11}$$

Note that the unit of the room absorption A is m^2.

In the steady state the sound power emitted by the source is counterbalanced by the sound power absorbed by the walls. The latter is simply the product of the incident sound power per unit area and the total absorption, so we finally obtain the equation

$$P_{\text{a,source}} = P_{\text{a,abs}} = I_{\text{inc}}A = \frac{p_{\text{rms}}^2}{4\rho c}A. \tag{3.12}$$

Equation (3.12) makes it possible to estimate the sound power of a source from the sound pressure it generates in a reverberation room. All that is required is the total absorption of the room. A more accurate version of this equation is derived in section 5.1.

The theory of the perfectly diffuse sound field assumes that the waves that constitute the sound field are uncorrelated. Therefore interference effects between the waves are ignored, and the mean square pressures of the various waves are regarded as additive. However, it is possible to extend the theory to take account of the interference phenomena that occur

9 In general the absorption coefficient of a given material depends on the nature of the sound field. Here we are concerned with the diffuse field (or random incidence) absorption coefficient.

near the walls of the room. The theory is due to Waterhouse [13]. Waterhouse assumed that an infinitely large, perfectly rigid plane surface was exposed to perfectly diffuse sound incidence.

Near the wall at $z = l_z$ eq. (3.5) should be modified to

$$\hat{p}(z,\theta,\varphi) = \frac{C}{\sqrt{4\pi}}\left(e^{-jk_z z} + e^{jk_z z}\right)e^{j(\omega t - k_x x - k_y y)} = \frac{C}{\sqrt{\pi}}\cos k_z z\, e^{j(\omega t - k_x x - k_y y)}, \qquad (3.13)$$

because each incident wave will be fully coherent with the corresponding reflected wave. The sound incidence is now restricted to a hemisphere, and eq. (3.7) becomes

$$\begin{aligned} p_{\mathrm{rms}}^2(z) &= \frac{|C|^2}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\varphi\int_0^{\pi/2}\cos^2(kz\cos\theta)\sin\theta\,\mathrm{d}\theta \\ &= \frac{|C|^2}{2}\int_0^{\pi/2}\left(1+\cos(2kz\cos\theta)\right)\sin\theta\,\mathrm{d}\theta = \frac{|C|^2}{2}\left(1+\frac{\sin 2kz}{2kz}\right), \end{aligned} \qquad (3.14)$$

which shows that the interference between each wave and its reflected counterpart gives rise to an interference pattern near the boundaries of the room; see figure 3.2. Note that the mean square sound pressure is doubled at the surface of the wall, corresponding to an increase of the level of 3 dB. Similar considerations show that there is a systematic increase of the sound pressure level of 6 dB near the edges of a reverberation room and an increase of 9 dB near the corners of a the room [13].

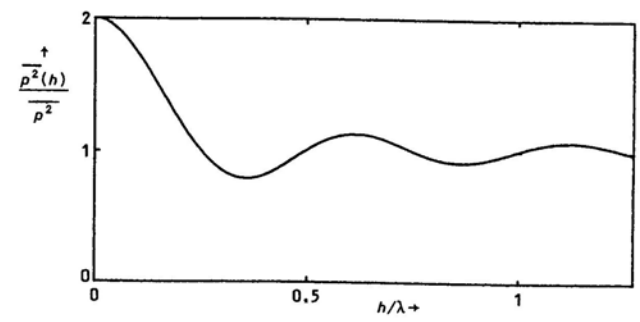

Figure 3.2 Interference pattern in a diffuse sound field in front of a rigid wall.

Waterhouse used eq. (3.14) to calculate the fractional increase of the total sound energy in the room due to the interference pattern near the walls [13]. Ignoring the interference effects near edges and corners he obtained the following approximate expression

$$\frac{S}{V}\int_0^{\infty}\frac{\sin(2kh)}{2kh}\mathrm{d}h = \frac{S}{2kV}\int_0^{\infty}\frac{\sin x}{x}\mathrm{d}x = \frac{\pi S}{4kV} = \frac{S\lambda}{8V}, \qquad (3.15)$$

where λ is the wavelength, and concluded that, instead of the total energy $V\overline{p^2}/(\rho c^2)$, which is what one would expect,[10] the total sound energy in the room amounts to

[10] In a resonant system the time averaged potential energy equals the time averaged kinetic energy. Therefore the total energy is simply twice the potential energy.

$$E_a \simeq V\frac{\overline{p^2}}{\rho c^2}\left(1+\frac{S\lambda}{8V}\right). \tag{3.16}$$

The factor in parenthesis is known as the ‘Waterhouse correction’. It is negligibly small except at low frequencies. In section 5.1 we shall use this quantity for improving eq. (3.12).

3.2 The sound field in a reverberation room driven with a pure tone

Exceedingly useful as it is, the perfectly diffuse field model is too coarse an approximation for some purposes. For example, it ignores the interference phenomena that are caused by the fact all the sound waves in the room are generated by the same source and postulates that the sound field is homogeneous (in obvious disagreement with the pure-tone expressions used in deriving eqs. (3.7), (3.10) and (3.12)), and this is not a good approximation in a reverberation room driven with a narrow-band signal. A more realistic model of the sound field in a reverberation room above the Schroeder frequency describes the sound field as composed of *coherent* plane waves with random phases arriving from all directions. This is a pure-tone model, and therefore the various plane waves interfere in the entire sound field, not just near the boundaries as assumed in section 3.1. As we shall see, the result is a sound field in which the sound pressure level depends on the position, although the *probability* of the level being in a certain interval is the same at all positions. Temporal correlation functions between linear quantities measured at two positions depend on the positions, although the *probability* of a given correlation function being in a certain interval depends only on the distance between the two points. The time-averaged sound intensity assumes a finite value at all positions. Since infinitely many plane waves with completely random phases are assumed, this model is also idealised, but it gives a good approximation to the sound field in a reverberation room driven with a pure tone above the Schroeder frequency. With averaging over an ensemble of realisations the perfectly diffuse field described above is obtained. An approximation to ensemble averaging is obtained if the room is equipped with a rotating diffuser.[11]

As mentioned above, the statistical model is based on the assumption that the sound field can be modelled as a sum of plane waves with random phases and amplitudes, arriving from random directions. Therefore the sound pressure at a given position can be written

$$\hat{p}(\mathbf{r}_1) = \lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} A_i \mathrm{e}^{\mathrm{j}(\omega t - \mathbf{k}_i\cdot\mathbf{r}_1)} \quad , \tag{3.17}$$

where A_i is the complex amplitude of the i’th wave and $\mathbf{k}_i$ is its wavenumber vector, that is, a vector of the length k (= ω/c) pointing in the direction of propagation of the i’th wave. The amplitudes are independent random variables with uniformly distributed phases, and all directions of the wavenumber vectors are equally probable. At another position in the sound field the sound pressure is

$$\hat{p}(\mathbf{r}_2) = \lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} A_i \mathrm{e}^{\mathrm{j}(\omega t - \mathbf{k}_i\cdot\mathbf{r}_2)} = \lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} A_i \mathrm{e}^{\mathrm{j}(\omega t - \mathbf{k}_i\cdot\mathbf{r}_1)}\mathrm{e}^{\mathrm{j}\mathbf{k}_i\cdot\mathbf{r}} \ , \tag{3.18}$$

where $\mathbf{r}$ is the vector that separates the two points. Thus $\mathbf{k}_i \cdot \mathbf{r}$ is a term that accounts for the phase shift between point no 1 and point no 2. Since the directions of the various waves are random this is a random term, and if $r > \lambda/2$ it takes random values over an interval of more than $\pm\pi$. In other words, at points more than half a wavelength apart we have in effect inde-

[11] A rotating diffuser is a large, slowly rotating device that changes the modal pattern in the room.

pendent sets of random phases. This leads to the conclusion that the statistical properties of the sound field can be determined by studying the *ensemble statistics* of sums of the type given by eq. (3.17).

The spatial statistics of the mean square pressure is particularly interesting, since spatial average values of this quantity are central in practically all measurements in reverberation rooms (cf. eq. (3.12)). The amount of spatial averaging that is required to ensure a reliable estimate is obviously of concern. The mean square pressure can be written

$$
\begin{aligned}
p_{\mathrm{rms}}^2(\mathbf{r}_1) &= \lim_{N\to\infty} \frac{1}{2N}\left|\sum_{i=1}^{N} A_i \mathrm{e}^{\mathrm{j}(\omega t-\mathbf{k}_i\cdot\mathbf{r}_1)}\right|^2 \\
&= \lim_{N\to\infty} \frac{1}{2N}\left|\sum_{i=1}^{N} |A_i|\left(\cos(\varphi_i-\mathbf{k}_i\cdot\mathbf{r}_1)+\mathrm{j}\sin(\varphi_i-\mathbf{k}_i\cdot\mathbf{r}_1)\right)\right|^2 \\
&= \lim_{N\to\infty} \frac{1}{2N}\left(\left(\sum_{i=1}^{N} |A_i|\cos(\varphi_i-\mathbf{k}_i\cdot\mathbf{r}_1)\right)^2+\left(\sum_{i=1}^{N} |A_i|\sin(\varphi_i-\mathbf{k}_i\cdot\mathbf{r}_1)\right)^2\right),
\end{aligned}
\tag{3.19}
$$

where φ_i is the phase of A_i. It can be seen that the mean square pressure is a sum of two squared sums of random terms. This is a random variable (see Appendix A), and from the central-limit theorem we conclude that each sum is a Gaussian variable (see Appendix B and ref. [14]). It can readily be shown that the two sums have the same statistical properties, and that they are independent random variables. A sum of n squared, independent Gaussian random variables with the same distribution has a chi-square distribution with n degrees of freedom (see Appendix C). In the particular case of n = 2 this is the *exponential distribution*, which leads to the conclusion that the mean square pressure in the room is a random variable with the probability density function

$$
f_{p_{\mathrm{rms}}^2}(x) = \begin{cases} \dfrac{1}{\xi}\mathrm{e}^{-x/\xi} & \text{if } x > 0 \\ 0 & \text{elsewhere,} \end{cases}
\tag{3.20}
$$

where

$$
\xi = E\{p_{\mathrm{rms}}^2\}
\tag{3.21}
$$

is the ensemble average of the mean square pressure, which is also the spatial average value. The exponential probability density is shown in figure 3.3. Equation (3.20) shows that the probability of the mean square pressure at a given position in the room exceeding a certain value ε can be written

$$
P\{p_{\mathrm{rms}}^2 > \varepsilon\} = \int_{\varepsilon}^{\infty} f_{p_{\mathrm{rms}}^2}(x)\mathrm{d}x = \frac{1}{E\{p_{\mathrm{rms}}^2\}}\int_{\varepsilon}^{\infty} \exp(-x/E\{p_{\mathrm{rms}}^2\})\mathrm{d}x\ .
\tag{3.22}
$$

One of the properties of an exponentially distributed random variable is that its relative standard deviation, that is, the standard deviation divided by the average value, equals unity; see Appendix C. From the foregoing we can now conclude that the mean square pressure in a reverberation room driven with a pure tone varies significantly with the position: its relative spatial standard deviation equals unity.

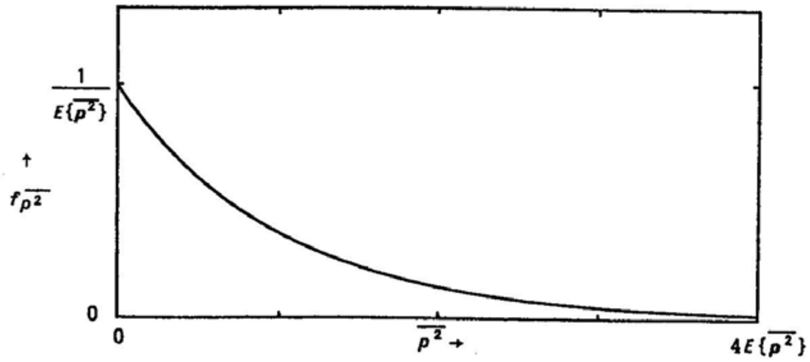

Figure 3.3 The exponential distribution.

It can be shown from eq. (3.20) that the probability density of the sound pressure *level* is [11]

$$f_{L_p}(x) = \frac{\ln 10}{10} \exp\left(\frac{\ln 10}{10}(x - L_0) - \exp\left(\frac{\ln 10}{10}(x - L_0) \right) \right), \quad (3.23)$$

where L_0 is the level that corresponds to $E\{p_{\text{rms}}^2\}$. This function is shown in figure 3.4. The corresponding standard deviation is about 5.6 dB. See also figure 3.5, which shows the sound pressure level recorded by a traversing microphone in a reverberation room.

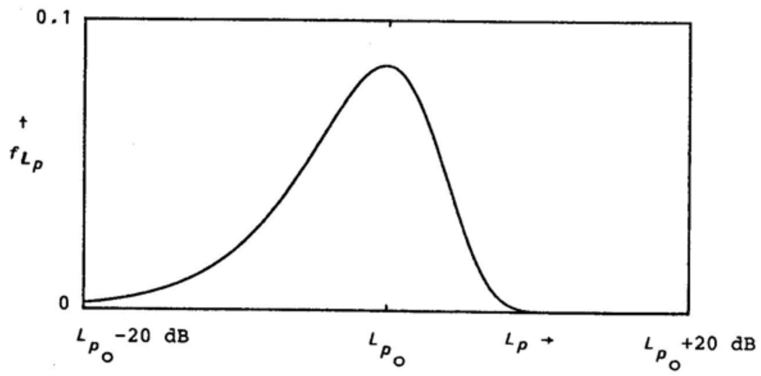

Figure 3.4 The probability density of the sound pressure level in a reverberation room driven with a pure tone.

Finally it is worth calling attention to the fact that no properties of the room have entered into these considerations – except a modal overlap of at least three. According to this theory the spatial standard deviation of the sound pressure level is about 5.6 dB in *any* lightly damped room driven with a pure tone above the Schroeder frequency except close to the source that generates the sound field.

It is physically obvious that the sound pressures at two points very close to each other tend to be similar; there is a limit to how rapidly the sound field changes with position. A statistical analysis of such a phenomenon leads to a description in terms of correlation functions. We can calculate the spatial correlation of the sound pressure as follows. Combining eqs. (3.17) and (3.18) gives

$$
\begin{aligned}
E\{\hat{p}(\mathbf{r}_1)\hat{p}^*(\mathbf{r}_2)\} &= E\left\{\lim_{N\to\infty}\frac{1}{N}\sum_i^N\sum_j^N\left|A_iA_j\right|\mathrm{e}^{\mathrm{j}(\varphi_i-\varphi_j)}\mathrm{e}^{-\mathrm{j}\mathbf{k}_i\cdot\mathbf{r}}\right\} \\
&= E\left\{\lim_{N\to\infty}\frac{1}{N}\sum_i\left|A_i\right|^2\cos(\mathbf{k}_i\cdot\mathbf{r})\right\} \\
&= E\left\{\left|p\right|^2\right\}\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\cos(kr\cos\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= E\left\{\left|p\right|^2\right\}\frac{1}{2kr}\int_{-kr}^{kr}\cos x\,\mathrm{d}x = E\left\{\left|p\right|^2\right\}\frac{\sin kr}{kr}.
\end{aligned}
\tag{3.24}
$$

The normalised correlation function is shown in figure 3.6. The interpretation of this spatial correlation function is that sound pressure signals recorded at positions less than, say, a quarter of a wavelength apart in a reverberation room driven with a pure tone are likely to have almost the same amplitude and phase. On the other hand, if there is more than half a wavelength between the two points then knowledge of the amplitude and phase of one of the signals gives practically no knowledge of the other signal.

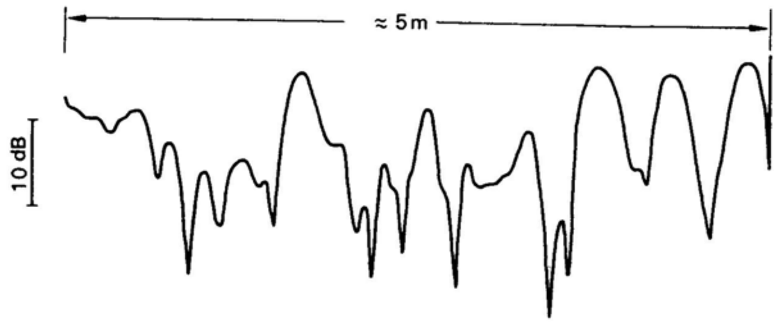

Figure 3.5 The sound pressure level as a function of the position on a diagonal path in a reverberation room driven with a pure tone. (From Kuttruff 2000.)

Using similar considerations one can show that the normalised spatial covariance of the mean square pressure is $(\sin kr/kr)^2$ [15]. This function is shown in figure 3.7. It is apparent that the covariance is negligible for distances exceeding half a wavelength, indicating that microphone positions should be spaced half a wavelength apart (or more) for maximum efficiency of the spatial averaging procedure. If the positions are less than half a wavelength apart they do not give independent sample values. With independent sample values the uncertainty of the resulting estimate is reduced by the square root of the number of positions.

Various continuous averaging procedures have been examined in refs. [16, 17]. One of the most important results is that averaging over a straight path of the length l is equivalent to averaging over $l/(\lambda/2)$ independent positions. In practice the spatial averaging is often carried out using a rotating microphone boom. If the diameter of the measurement path exceeds one wavelength then the equivalent number of independent positions may be calculated as if it were straight.

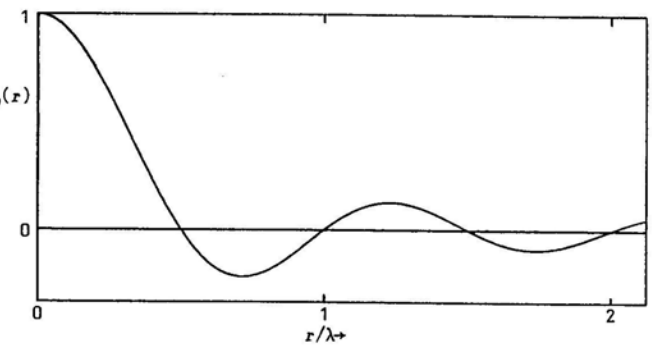

Figure 3.6 The spatial correlation of the sound pressure in a reverberation room driven with a pure tone.

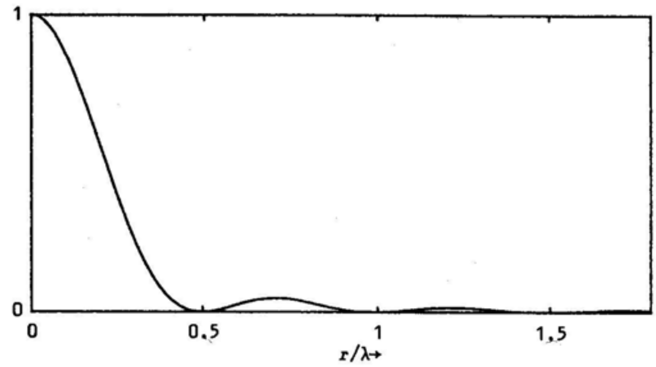

Figure 3.7 The spatial covariance of the mean square sound pressure in a reverberation room driven with a pure tone.

3.3 Frequency averaging

It is possible to extend the statistical pure-tone model to excitation with a band of noise. If a reverberation room is driven with a pure tone whose frequency is shifted slightly, the phases and amplitudes of the plane waves that compose the sound field are changed, which means that the entire interference pattern is changed [11]. The longer the reverberation time of the room the faster the sound field will change as a function of the frequency [18]. It now follows that excitation with a band of noise corresponds to averaging over the band. As a result the sound field becomes more uniform, temporal correlation functions between linear quantities measured at pairs of positions tend to depend less on the particular positions, the sound intensity is reduced, and the sound power of a monopole approaches the free field power. The effect of this spectral averaging depends not only on the bandwidth of the excitation (or the analysis) but also on the damping of the room; the longer the reverberation time the more efficient the averaging.

It can be shown [19, 20] that the relative spatial standard deviation of the mean square pressure in a reverberation room driven with a band of noise (or driven with wide band noise and analysed in bands) to a good approximation is given by the expression

$$\varepsilon\{\overline{p^2}\} \simeq \frac{1}{\sqrt{1+(\tau\Delta\omega/\pi)}} \simeq \frac{1}{\sqrt{1+\Delta f T_{60}/6.9}}, \tag{3.25}$$

where $\Delta\omega = 2\pi\Delta f$ is the bandwidth of the noise. This will usually be much less that the relative standard deviation of unity found in a room driven with a pure tone.

It is apparent that there are many similarities between the sound field in a room driven with noise and the perfectly diffuse sound field described in section 3.1; this explains the usefulness of this concept. However, there are also important differences between the 'fine structure' of the sound field in a real room and a perfectly diffuse sound field; this can be observed when the sound field is analysed with fine spectral resolution. It can also be concluded that diffuseness at low frequencies requires a large room, and that long reverberation times (within the limits determined by the requirement of sufficient modal overlap) are favourable.

3.4 The sound power emitted by a point source

The sound power emitted by a point source in a lightly damped room cannot be expected to be unaffected by the reverberant sound field even when it is placed far from the walls of the room. The sound power of a pure-tone monopole with the volume velocity $Qe^{j\omega t}$ is

$$P_a = \frac{|Q|^2}{2}\mathrm{Re}\{Z_r\}, \tag{3.26}$$

where Z_r is the radiation impedance. The sound field at the source position may be regarded as the sum of the direct field and the reverberant field, and therefore the radiation impedance is the sum of the free-field radiation impedance and the complex ratio of the sound pressure associated with the reverberant field and the volume velocity of the source,

$$Z_r = \frac{\rho c k^2}{4\pi} + \frac{\hat{p}_{\mathrm{rev}}}{Qe^{j\omega t}}. \tag{3.27}$$

All phases are equally probable in the reverberant field, and thus $E\{\hat{p}_{\mathrm{rev}}\} = 0$, which leads to the conclusion that *on the average* the monopole emits its free-field sound power output

$$E\{P_a\} = \frac{\rho c k^2 |Q|^2}{8\pi}. \tag{3.28}$$

However, the actual sound power output of the source varies with the position. The corresponding spatial variance can be calculated as follows,

$$\begin{aligned} \sigma^2\{P_a\} &= \frac{|Q|^4}{4}\sigma^2\{\mathrm{Re}\{Z_r\}\} = \frac{|Q|^4}{4}\sigma^2\left\{\mathrm{Re}\left\{\frac{\hat{p}_{\mathrm{rev}}}{Qe^{j\omega t}}\right\}\right\} = \frac{|Q|^2}{4}E\left\{|\hat{p}_{\mathrm{rev}}|^2\cos^2(\varphi)\right\} \\ &= \frac{|Q|^2}{8}E\left\{|\hat{p}_{\mathrm{rev}}|^2\right\} \simeq \frac{|Q|^2}{8}\frac{8\rho c}{A}E\{P_a\} \simeq E^2\{P_a\}\frac{8\pi}{k^2 A}, \end{aligned} \tag{3.29}$$

where use has been made of eqs. (3.12) and (3.28) and the fact that the average of a squared cosine of a random phase is ½. It can now be seen that the relative standard deviation is inversely proportional to the modal overlap,

$$\varepsilon\{P_a\} \simeq \frac{1}{k}\sqrt{\frac{8\pi}{A}} = \sqrt{\frac{1}{M\pi}} \tag{3.30}$$

(cf. eq. (3.3)). However, this derivation has not taken account of the phenomenon known as 'weak Anderson localisation' (also known as 'coherent backscattering'), according to which there is a concentration of the *reverberant* part of the sound field exactly at the source position [21]. This effect increases the relative standard deviation by a factor of $\sqrt{2}$. The resulting relation is compared with numerical data in figure 3.8.

It is apparent that the sound power emitted by the source at low frequencies varies substantially with room and position unless the room is very large and heavily damped. This indicates that it is extremely important to average over many source positions if the sound power of a source with a significant content of pure tones is to be determined in a reverberation room. Ideally one should also average over several rooms.

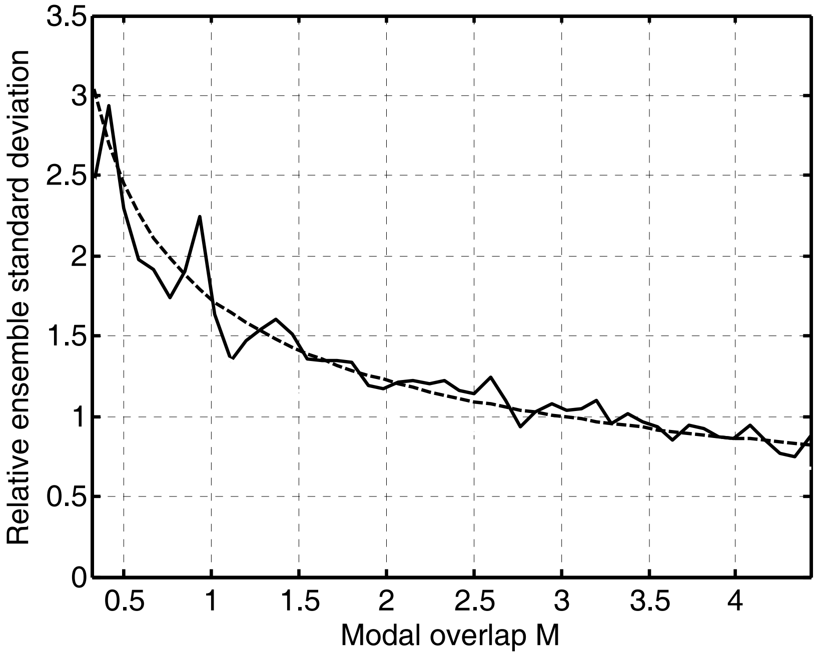

Figure 3.8 Relative ensemble standard deviation of the sound power output of a monopole that emits a pure tone (eq. (3.30)) compared with numerical (finite element) results. (From ref. [22]).

If the source emits bandpass filtered noise the spatial standard deviation is reduced by the factor given by eq. (3.25). Thus

$$\varepsilon\{P_{\mathrm{a}}\} \simeq \frac{1}{k_0}\sqrt{\frac{16\pi}{A}}\frac{1}{\sqrt{1+(\tau\Delta\omega/\pi)}} = \frac{1}{k_0}\sqrt{\frac{16\pi}{A}}\frac{1}{\sqrt{1+8V\Delta f/cA}} \simeq \frac{1}{k_0}\sqrt{\frac{2\pi c}{V\Delta f}}, \qquad (3.31)$$

where k_0 corresponds to the centre frequency. In this case the damping of the room has almost no influence. It can be seen that although the standard deviation of the emitted sound power is significantly reduced by the frequency averaging it is still not negligible at low frequencies unless the room is very large. It must be concluded that there is a considerable uncertainty in sound power measurements and in predictions based on sound power at low frequencies.

Extending the random wave model to the region below the Schroeder frequency

The model presented in section 3.2 is only valid when the modal overlap of the room is high. In this frequency range the source emits essentially its free field sound power; cf. eq. (3.30). However, it is possible to extend the model to lower frequencies by taking account of the variations of the sound power of the source. These random variations modify the ensemble average of the amplitudes of the waves. The result is an additional contribution to the variance of the mean square pressure that is inversely proportional to the modal overlap [23].

Statistical results based on the modal theory

It is possible to derive expressions in closed form for statistical properties of the sound power output of the source and the mean square pressure in the room by averaging expressions determined from eq. (2.40) over source and/or receiver position and replacing the resulting modal sums by integrals [24-26]. The latter procedure may be interpreted as determining averages over an ensemble of rooms with slightly different dimensions but essentially the same size. Such rooms would have the same modal density, but the particular distributions of modal frequencies would be different.

The theory is fairly complicated because the modal frequencies tend to exhibit 'long range repulsion', and thus they are not independently distributed on the frequency axis with a density given by eq. (2.21) as originally though; they are distributed in accordance with the random matrix theory of Gaussian orthogonal ensembles [27]. This theory, which is now well established, gives results in good agreement with the recently extended random wave theory; see ref. [23].

4. THE DECAY OF SOUND IN A LIGHTLY DAMPED ROOM

Consider a room driven by a source that emits sound, but not necessarily steady sound. Simple energy balance considerations lead to the following equation,

$$P_{\mathrm{a,source}} - P_{\mathrm{a,abs}} = \frac{\mathrm{d}E_{\mathrm{a}}}{\mathrm{d}t}, \tag{4.1}$$

where $P_{\mathrm{a,source}}$ is the sound power emitted by the source, $P_{\mathrm{a,abs}}$ is the sound power absorbed by the walls of the room, and E_{a} is the total sound energy in the room. (Obviously, eq. (4.1) is an extension of eq. (3.12)). If the source is suddenly turned off eq. (4.1) becomes

$$P_{\mathrm{a,abs}} + \frac{\mathrm{d}E_{\mathrm{a}}}{\mathrm{d}t} = 0, \tag{4.2}$$

which is the basis for the following simple considerations. We will study the problem using the two fundamentally different models presented in chapters 2 and 3.

4.1 The modal approach

In this section we will study the sound field mode by mode. The walls of the room have a finite but small admittance with a finite real part, indicating that acoustic energy is absorbed by the walls. The following considerations are based on the assumption that the mode shapes in the lightly damped room are the same as in the undamped room. It is also assumed that each mode maintains its shape during the decay process; it is just the amplitude that decreases with time.

We can express the total sound energy in the mode in terms of an integral of the squared mode shape as follows,

$$E_{\mathrm{a},m} = \int_V (w_{\mathrm{kin},m} + w_{\mathrm{pot},m})\mathrm{d}V = \int_V 2w_{\mathrm{pot},m}\mathrm{d}V = \int_V \frac{|\hat{p}_m|^2}{2\rho c^2}\mathrm{d}V = \int_V \frac{A_m^2\psi_m^2}{2\rho c^2}\mathrm{d}V, \tag{4.3}$$

where A_m is the amplitude of the mode (cf. eq. (2.12)). The sound power absorbed by the walls is obtained by integrating the product of the mean square sound pressure and the real part of the local admittance over the surface that defines the room,

$$P_{\mathrm{a,abs},m} = \int_S \mathbf{I}\cdot\mathrm{d}\mathbf{S} = \int_S \frac{|\hat{p}_m|^2}{2}\mathrm{Re}\{Y\}\mathrm{d}S = \int_S \frac{A_m^2\psi_m^2}{2}\mathrm{Re}\{Y\}\mathrm{d}S, \tag{4.4}$$

where Y is the local specific wall admittance (the local ratio of the normal component of the particle velocity to the sound pressure).

Under the reasonable assumption that the mode shape is maintained during the decay process we can expect the ratio $E_{\mathrm{a}} / P_{\mathrm{a,abs}}$ associated with a given mode to be constant. Apparently this ratio has the dimension of time,

$$\tau_m = \frac{E_{\mathrm{a},m}}{P_{\mathrm{a,abs},m}} = \frac{1}{\rho c^2} \frac{\int_V \psi_m^2 \mathrm{d}V}{\int_S \psi_m^2 \mathrm{Re}\{Y\} \mathrm{d}S}. \tag{4.5}$$

We can now write eq. (4.2) in the form

$$\frac{E_{\mathrm{a},m}}{\tau_m} + \frac{\mathrm{d}E_{\mathrm{a},m}}{\mathrm{d}t} = 0. \tag{4.6}$$

The solution to this simple first-order differential equation is a decaying exponential,

$$E_{\mathrm{a},m}(t) = E_{\mathrm{a},m}(t_0) \mathrm{e}^{-(t-t_0)/\tau_m} \qquad \text{for} \quad t \ge t_0, \tag{4.7}$$

where t_0 is the time at which the source is switched off. We conclude that the sound energy in each mode decreases exponentially with a time constant given by eq. (4.5).

The decay constants of the modes in a rectangular room with uniform wall admittance can be found by combining eqs. (2.11) and (4.5). Because of the normalisation, the volume integral in the numerator of eq. (4.5) simply equals V (cf. eq. (2.28)). The surface integral of the denominator becomes

$$\int_S \psi_m^2 \mathrm{Re}\{Y\} \mathrm{d}S \simeq \begin{cases} 8 \mathrm{Re}\{Y\} S \dfrac{1}{4} & \text{for oblique modes,} \\ 4 \mathrm{Re}\{Y\} S \left(\dfrac{1}{3}\dfrac{1}{4} + \dfrac{2}{3}\dfrac{1}{2} \right) & \text{for tangential modes,} \\ 2 \mathrm{Re}\{Y\} S \left(\dfrac{1}{3} + \dfrac{2}{3}\dfrac{1}{2} \right) & \text{for axial modes,} \end{cases} \tag{4.8}$$

where the first factor (8, 4 or 2) is the square of the normalisation constant given by eq. (2.14). The equation is exact for oblique modes; for axial and tangential modes it has been assumed that $l_x \simeq l_y \simeq l_z$ so that each wall has an area of $S/6$. Inserting in eq. (4.5) gives

$$\tau_m \simeq \begin{cases} \dfrac{V}{2\rho c^2 S \mathrm{Re}\{Y\}} = \dfrac{V}{2cS \mathrm{Re}\{\beta\}} & \text{for oblique modes,} \\ \dfrac{3V}{5\rho c^2 S \mathrm{Re}\{Y\}} = \dfrac{3V}{5cS \mathrm{Re}\{\beta\}} & \text{for tangential modes,} \\ \dfrac{3V}{4\rho c^2 S \mathrm{Re}\{Y\}} = \dfrac{3V}{4cS \mathrm{Re}\{\beta\}} & \text{for axial modes,} \end{cases} \tag{4.9}$$

where for simplicity we have introduced the normalised (dimensionless) specific wall admittance

$$\beta = \rho c Y. \tag{4.10}$$

It is apparent that axial modes die out at a slightly slower rate than tangential and oblique modes. In practice the losses of rooms are measured in frequency bands, say, one-third octave bands, which means that many modes are excited at the same time. As a result of the different decay rates logarithmic decay functions recorded in rectangular rooms without diffusing plates[12] tend to be curved at low frequencies; see figure 4.1. At higher frequencies oblique modes dominate, and the logarithmic decay functions tend to be almost perfectly linear.

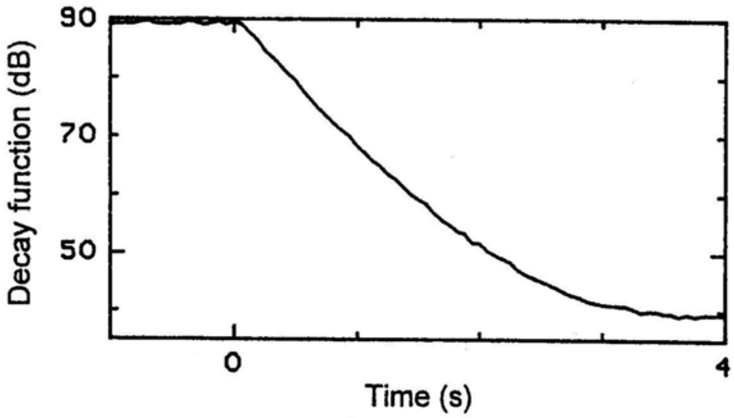

Figure 4.1 A curved decay curve recorded in a reverberation room. (From ref. [28].)

If we consider a rectangular room in which the losses are concentrated on one of the walls, say $z = 0$, a similar analysis gives

$$\int_S \psi_m^2 \operatorname{Re}\{Y\}\, dS = \begin{cases} 8\operatorname{Re}\{Y\} l_x l_y \frac{1}{4} & \text{for oblique modes,} \\ 4\operatorname{Re}\{Y\} l_x l_y \frac{1}{4} & \text{if } n_x > 0 \text{ and } n_y > 0 \text{ and } n_z = 0, \\ 4\operatorname{Re}\{Y\} l_x l_y \frac{1}{2} & \text{if } n_x = 0 \text{ and } n_y > 0 \text{ and } n_z > 0, \\ 4\operatorname{Re}\{Y\} l_x l_y \frac{1}{2} & \text{if } n_x > 0 \text{ and } n_y = 0 \text{ and } n_z > 0, \\ 2\operatorname{Re}\{Y\} l_x l_y & \text{if } n_x = 0 \text{ and } n_y = 0 \text{ and } n_z > 0, \\ 2\operatorname{Re}\{Y\} l_x l_y \frac{1}{2} & \text{if } n_x > 0 \text{ and } n_y = 0 \text{ and } n_z = 0, \\ 2\operatorname{Re}\{Y\} l_x l_y \frac{1}{2} & \text{if } n_x = 0 \text{ and } n_y > 0 \text{ and } n_z = 0, \end{cases} \tag{4.11}$$

[12] Diffusers in the form of large stationary plates distributed at random in the room and having random orientation tend to couple the various modes and thus make the decay curves more linear.

which leads to the conclusion that there are only two different values of τ_m, depending on whether there is wave motion in the z-direction or not,

$$\tau_m = \begin{cases} \dfrac{l_z}{2c\,\mathrm{Re}\{\beta\}} & \text{for } n_z \neq 0, \\ \dfrac{l_z}{c\,\mathrm{Re}\{\beta\}} & \text{for } n_z = 0. \end{cases} \tag{4.12}$$

Note that the two decay constants differ by a factor of two, which is more than in the room with uniform absorption. In rooms with dominating absorption on just one surface the decay curves tend to be more curved than in rooms with a more uniform distribution of the losses.

4.2 The statistical approach

Assuming a stationary ideal diffuse sound field in the room we can express the total sound energy in the room as

$$E_a = V \frac{< p_{\mathrm{rms}}^2 >}{\rho c^2}, \tag{4.13}$$

where $< p_{\mathrm{rms}}^2 >$ is the spatial average of the mean square sound pressure The absorbed sound power is

$$P_{\mathrm{a,abs}} = \int_S \mathbf{I} \cdot \mathrm{d}\mathbf{S} = \int_S I_{\mathrm{inc}} \alpha \, \mathrm{d}S = \frac{< p_{\mathrm{rms}}^2 >}{4\rho c} A \tag{4.14}$$

(cf. eq. (3.12)). Note that this is *not* a mode-by-mode analysis; by contrast we must assume excitation with a band of noise (cf. the considerations in section 3.3). Under the reasonable assumption that the two expressions given by eqs. (4.13) and (4.14) remain valid also during the decay process[13] we can now introduce the time constant τ, defined as the ratio

$$\tau = \frac{E_{\mathrm{a}}}{P_{\mathrm{a,abs}}} = \frac{4V}{cA}. \tag{4.15}$$

Assuming uniform absorption at the walls (which implies that $A = S\alpha$) and comparing eq. (4.9) (for three-dimensional modes) and eq. (4.15) leads to the conclusion that they agree if

$$\alpha = 8\,\mathrm{Re}\{\beta\}. \tag{4.16}$$

To first order this result is in agreement with what is found by calculating the dissipated fraction of the incident sound power per unit area of an infinite plane surface with the normalised specific admittance β, assuming uniform sound incidence (see, e.g., Morse and Ingard, 1968).

If only the wall at $z = 0$ is absorbing eq. (4.14) becomes

[13] Obviously p_{rms}^2 must be interpreted as a running short-time average of the squared sound pressure for this to be meaningful.

$$P_{\mathrm{a,abs}} = \int_S \mathbf{I}\cdot \mathrm{d}\mathbf{S} = \frac{< p_{\mathrm{rms}}^2 >}{4\rho c} l_x l_y \alpha \,, \tag{4.17}$$

and eq. (4.15) becomes

$$\tau = \frac{E_{\mathrm{a}}}{P_{\mathrm{a,abs}}} = \frac{4V}{c l_x l_y \alpha} = \frac{4 l_z}{c\alpha}, \tag{4.18}$$

and this is seen to be in agreement with eq. (4.12) for the same relation between α and $\mathrm{Re}\{\beta\}$, confirming the consistency of the modal and the statistical theory.

5. APPLICATIONS OF REVERBERATION ROOMS

Reverberation rooms are used for a number of standardised measurements, some of the most important of which are briefly described in the following.

5.1 Sound power determination

Reverberation rooms are useful for measurement of sound power, in particular the sound power of machines that operate in long cycles, since the alternatives, the sound intensity method and the free field method, are not very suitable for such sources.. The considerations presented in sections 3.3 and 3.4 lead to the conclusions that to reduce the uncertainty associated with sound power measurements of sources that emit pure tones the reverberation room should be large and fairly damped; whereas the best facilities for measurements of the sound power of sources that essentially emit random noise are large reverberation rooms with a long reverberation time.

In section 3.1 a relation between the sound power emitted by a source in a reverberation room, the sound pressure generated by the source, and the total room absorption was derived (eq. (3.12)) on the basis of considerations that made use of the concept of a perfectly diffuse sound field. We should now be able to develop a more precise relation.

From eqs. (4.1) and (4.2) we conclude that the sound power emitted by the source in the steady state must be identical with the absorbed sound power immediately after the source has been switched off,

$$\begin{aligned} P_{\mathrm{a,source}} &= -\left.\frac{\mathrm{d}E_{\mathrm{a}}}{\mathrm{d}t}\right|_{t=t_0} = -\left.\frac{E_{\mathrm{a}}(t_0)\mathrm{d}(\ln(E_{\mathrm{a}}(t)/E_0))}{\mathrm{d}t}\right|_{t=t_0} \\ &= -\frac{\ln 10}{10}\left.\frac{E_{\mathrm{a}}(t_0)\mathrm{d}(10\log(E_{\mathrm{a}}(t)/E_0))}{\mathrm{d}t}\right|_{t=t_0} = -\frac{< p_{\mathrm{rms}}^2 > V}{\rho c^2}\left(1+\frac{S\lambda}{8V}\right)\frac{\ln 10}{10} d_{\mathrm{e}} \\ &= \frac{< p_{\mathrm{rms}}^2 > V}{\rho c^2}\left(1+\frac{S\lambda}{8V}\right)\frac{6\ln 10}{T_{\mathrm{e}}} = \frac{< p_{\mathrm{rms}}^2 > V}{\rho c^2}\left(1+\frac{S\lambda}{8V}\right)\frac{13.8}{T_{\mathrm{e}}}, \end{aligned} \tag{5.1}$$

where d_{e} is the initial rate of decay of the logarithmic decay curve (in dB/s) and

$$T_{\mathrm{e}} = -\,60 / d_{\mathrm{e}} \tag{5.2}$$

is the corresponding reverberation time.[14] Note that the Waterhouse correction (eq. (3.16)) has been applied to account for the additional sound energy near the boundaries of the room.[15]

Since, by definition,

$$10\log\left(\mathrm{e}^{-T_{60}/\tau}\right) = -60 \quad (\mathrm{dB}), \tag{5.3}$$

we have the following relation between the time constant and the reverberation time,

$$T_{60} = 6\ln 10\ \tau \simeq 13.8\ \tau, \tag{5.4}$$

which shows that eq. (5.1) is in agreement with eq. (4.15) (apart from the Waterhouse correction) when the decay curves are linear.

5.2 Measurement of sound absorption

Reverberation rooms are also used for measurement of the sound absorption of acoustic materials. In the standardised method of determining diffuse-field sound absorption a large sample of the material under test ($10\ \mathrm{m}^2$) is installed in the room, and the absorption area and thus the absorption coefficient is deduced from the resulting reduction of the reverberation time,

$$\alpha = \frac{4V \cdot 6\ln 10}{cS_\mathrm{s}}\left(\frac{1}{T_{60}^\mathrm{s}} - \frac{1}{T_{60}}\right) \simeq \frac{55.3V}{cS_\mathrm{s}}\left(\frac{1}{T_{60}^\mathrm{s}} - \frac{1}{T_{60}}\right), \tag{5.5}$$

where T_{60}^s is the reverberation time with the absorbing specimen in the room, T_{60} is the reverberation time without the material in the room, and S_s is the area of the specimen. However. it is well known that this method gives results that vary rather significantly from one room to another even under the conditions that are specified in the standard [30, 31], which shows that the simple statistical theory based on eqs. (3.5) and (4.15) is, after all, only approximate.

5.3 Measurement of transmission loss

A suite of reverberation rooms can be used for measurement of the transmission loss of the partition between two adjoining rooms provided that (unwanted) flanking transmission is negligible. In the source room the sound power incident on the partition under test is deduced from the spatial average of the mean square pressure (cf. eq. (3.10)); and in the receiver room the transmitted sound power is determined as described in section 5.1. The transmission loss is the ratio of incident to transmitted sound power (in decibels).

In the receiving room one should obviously use the Waterhouse correction, although this is not recommended by the measurement standard (for political reasons) [32]. In the source room one should use a similar correction, as shown only recently [33].

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[14] In practice an interval of about 10 dB is used when the 'early decay time' is determined from reverberant decay functions, so T_e is the time it takes for the level to decrease 10 dB multiplied by a factor of 6. At medium and high frequencies the logarithmic decay curves are usually almost perfectly linear, and then the interval does not matter.

[15] The Waterhouse correction is negligible except at low frequencies. Since Waterhouse's derivation was based on an assumption of ideal diffuse sound incidence, one might expect the resulting 'correction' to be less accurate at low frequencies, which is where it really matters. However, much later numerical calculations have confirmed the validity of eq. (3.16) [29].

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Sound in enclosures is an enormously wide field ranging from sound in small acoustic couplers for calibrating microphones to the acoustics of auditoria and concert halls. Relatively introductory treatments of various aspects of this wide topic are found in the books by Kinsler *et al*., by Fahy, and by Morse. Kuttruff's book deals more thoroughly with room acoustics. His chapter in Encyclopedia of Acoustics is an abbreviated version of the same text. A very comprehensive treatment of architectural acoustics is given in the book by Cremer and Müller. The book by Morse and Ingard is recommended for its treatment of the modal theory. Pierce's book is recommended for its exposition of the statistical theory.

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APPENDIX A: RANDOM VARIABLES

Random (or stochastic) variables can be described in terms of their probability density functions. By definition the integral of such a function from minus infinity to infinity is unity,

$$P\left\{-\infty < x < \infty\right\} = \int_{-\infty}^{\infty} f_x(u)\mathrm{d}u = 1\,, \tag{A1}$$

indicating that the variable must always assume some value. The integral of the probability density over a certain interval gives the probability of the variable being in that interval. Thus

$$P\left\{a \le x \le b\right\} = \int_{a}^{b} f_x(u)\mathrm{d}u \tag{A2}$$

is the probability that the random variable *x* with the probability density f_x takes a value in the interval from *a* to *b*.

It is sometimes sufficient to characterise a random variable in terms of its average (or mean or expected) value and its variance or standard deviation. The average value of *x* is

$$E\left\{x\right\} = \int_{-\infty}^{\infty} u f_x(u)\mathrm{d}u\,. \tag{A3}$$

The variance is

$$\sigma^2\left\{x\right\} = E\left\{\left(x - E\left\{x\right\}\right)^2\right\} = \int_{-\infty}^{\infty} \left(u - E\left\{x\right\}\right)^2 f_x(u)\mathrm{d}u\,. \tag{A4}$$

The square root of the variance, σ, is called the standard deviation. It is often useful to normalise the standard deviation with the average value,

$$\varepsilon\left\{x\right\} = \sigma\left\{x\right\} / E\left\{x\right\}. \tag{A5}$$

APPENDIX B: THE CENTRAL LIMIT THEOREM

The central limit theorem states that a sum of independent random variables having the same probability distribution tends to become normally distributed, that is, the probability function of the sum

$$z = x_1 + x_2 + \cdots + x_n \tag{B1}$$

tends to

$$f_z(u) \simeq \frac{1}{\sigma\sqrt{2}} e^{-(u-\mu)^2/2\sigma^2}\,, \tag{B2}$$

where

$$\mu = n \times E\left\{x_i\right\} \tag{B3}$$

and

$$\sigma = \sqrt{n \times \sigma^2\left\{x_i\right\}}\,. \tag{B4}$$

Actually it is not even necessary for the independent random variables to have the same probability distribution for the sum to become normally distributed; see ref. [14].

APPENDIX C: Chi and chi-square statistics

Let x_1, x_2, ...x_n be independent normally distributed (or Gaussian) variables with zero mean, variance σ^2 and the probability density

$$f_{x_i}(u) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{u^2}{2\sigma^2}\right), \tag{C1}$$

corresponding to the joint probability function

$$f(u_1, u_2, \ldots, u_n) = \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \exp\left(-\frac{u_1^2 + u_2^2 + \cdots + u_n^2}{2\sigma^2}\right). \tag{C2}$$

We now define two random variables,

$$\chi_n = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \tag{C3}$$

and

$$\chi_n^2 = x_1^2 + x_2^2 + \ldots + x_n^2 . \tag{C4}$$

They are known as chi and chi-square statistics with n degrees of freedom, respectively, and it can be shown that they have the following probability density functions [14],

$$f_{\chi_n}(u) = \begin{cases} 0 & \text{for}\, u < 0 \\ \dfrac{2}{\left(\sqrt{2}\sigma\right)^n \Gamma(n/2)} u^{n-1} \mathrm{e}^{-u^2/2\sigma^2} & \text{for}\ u \geq 0, \end{cases} \tag{C5}$$

$$f_{\chi_n^2}(u) = \begin{cases} 0 & \text{for}\, u < 0 \\ \dfrac{u^{n/2-1}}{\left(\sqrt{2}\sigma\right)^n \Gamma(n/2)} \mathrm{e}^{-u/2\sigma^2} & \text{for}\ u \geq 0, \end{cases} \tag{C6}$$

where Γ is the gamma function,

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t . \tag{C7}$$

If the argument of the gamma function is a positive integer m, then

$$\Gamma(m) = (m-1)! = 1 \times 2 \times \cdots \times (m-1) . \tag{C8}$$

In the particular case where n = 2 eqs. (C5) and (C6) become the Rayleigh probability density,

$$f_{\chi_2}(u) = \begin{cases} 0 & \text{for}\, u < 0 \\ \dfrac{u}{\sigma^2} \mathrm{e}^{-u^2/2\sigma^2} & \text{for}\ u \geq 0, \end{cases} \tag{C9}$$

and the exponential probability density,

$$f_{\chi_2^2}(u) = \begin{cases} 0 & \text{for } u < 0 \\ \dfrac{e^{-u/2\sigma^2}}{2\sigma^2} & \text{for } u \geq 0, \end{cases} \tag{C10}$$

respectively; see figure C1. It is easy to show that a Rayleigh distributed random variable has a relative standard deviation of

$$\varepsilon\left\{\chi_2\right\} = \sqrt{\frac{4-\pi}{\pi}} \simeq 0.52\,, \tag{C12}$$

whereas an exponentially distributed variable has a relative standard deviation of unity:

$$\varepsilon\left\{\chi_2^2\right\} = 1\,. \tag{C13}$$

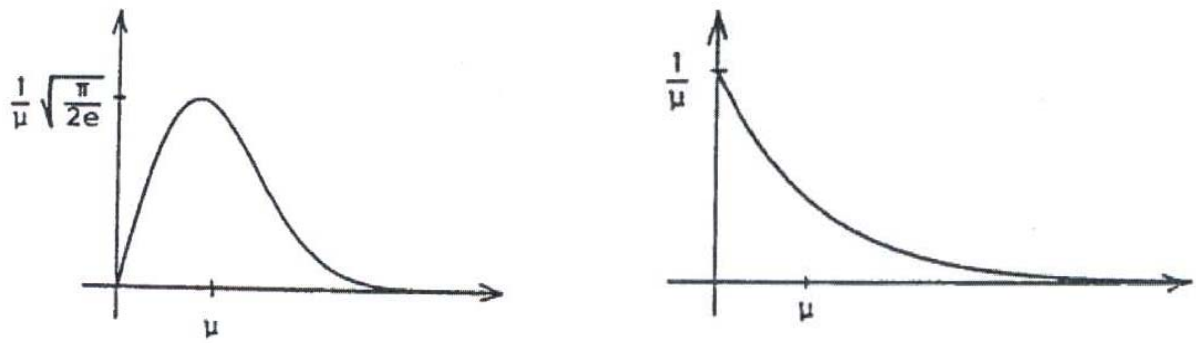

Figure C1. Left: Rayleigh distribution; right: exponential distribution. In both cases the parameter μ is the mean value.

LIST OF SYMBOLS

A	room absorption [m^2]
c	speed of sound [m/s]
d_e	initial rate of decay [dB/s]
E_a	sound energy in a room [J]
$E\{\ \}$	expected value
f	frequency [Hz]
f_s	Schroeder frequency [Hz]
$f_x(y)$	probability density function
G	Green's function [m^{-1}]
h	distance from wall [m]
$\mathbf{I}$	sound intensity [W/m^2]
I_{inc}	incident sound power per unit area [W/m^2]
k	wavenumber [m^{-1}]
k_x	component of wavenumber [m^{-1}]
$\mathbf{k}_i$	random wavenumber vector [m^{-1}]
l_x, l_y, l_z	dimensions of rectangular room [m]
m	integer
M	modal overlap [dimensionless]
$n(f)$	modal density [Hz^{-1}]
$N(f)$	number of modes below f [dimensionless]
n_x	integer
p	sound pressure [Pa]
p_0	static pressure [Pa]
p_{rms}	rms (root mean square) sound pressure [Pa]
$\hat{p}$	sound pressure (complex representation) [Pa]
P_a	sound power [W]
$P_{a,abs}$	absorbed sound power [W]
$P\{\ \}$	probability
Q	volume velocity of monopole [m^3/s]
r	distance [m]
$\mathbf{r}$	receiver position [m]
$\mathbf{r}_0$	source position [m]
S	surface area [m^2]
S_s	area of absorbing specimen [m^2]
t	time [s]
T_e	reverberation time calculated from the initial part of the decay [s]
T_{60}	reverberation time [s]
u_x	particle velocity component [ms^{-1}]
V	volume [m^3]
w_{kin}	kinetic energy density [kgm^{-2}s^{-2}]
w_{pot}	potential energy density [kgm^{-2}s^{-2}]
x, y, z	Cartesian coordinates [m]
Y	specific wall admittance [kg^{-1}m^{2}s]
α	absorption coefficient [dimensionless]
β	normalised wall admittance [dimensionless]
γ	ratio of specific heats [dimensionless]
Γ	gamma function [dimensionless]

δ_{ij} Kronecker symbol (0 if $i \neq j$; 1 if $i = j$) [dimensionless]
$\delta(\mathbf{r})$ spatial delta function [m^{-3}]
$\varepsilon\{\ \}$ relative standard deviation
ε_i Neumann symbol (1 if $i = 0$; 2 if $i > 0$) [dimensionless]
Δf 3-dB bandwidth of mode [Hz]
θ polar angle in spherical coordinate system [radian]
λ wavelength [m]
Λ_N normalisation constant [dimensionless]
ρ density of air [kgm^{-3}]
$\sigma^2\{\ \}$ variance
τ time constant [s]
τ_m modal time constant [s]
φ azimuth angle in spherical coordinate system [radian]
φ_i random phase angle [radian]
ψ_m mode function [dimensionless]
ω angular frequency [radian/s]
ω_N natural angular frequency of mode [radian/s]

INDEX